

# POINCARÉ PROBLEM FOR WEIGHTED PROJECTIVE FOLIATIONS

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**ABSTRACT.** We give a bounding of degree of quasi-smooth hypersurfaces which are invariant by a one dimensional holomorphic foliation of a given degree on a weighted projective space.

## 1. INTRODUCTION

Henri Poincaré studied in [12] the problem to decide if a holomorphic foliation  $\mathcal{F}$  on the complex projective plane  $\mathbb{P}^2$  admits a rational first integral. Poincaré observed that in order to solve this problem is sufficient to find a bound for the degree of the generic curve invariant by  $\mathcal{F}$ . In general, this is not possible, but doing some hypothesis we obtain an affirmative answer for this problem, which nowadays is known as *Poincaré Problem*. This problem was treated by D. Cerveau and A. Lins Neto [5]. M. Brunella in [2] observed that obstruction to the positive solution to Poincaré problem is given by the GSV index. There exist several works about Poincaré problem and its generalizations, see for instance the papers: M. Carnicer [4], J. V. Pereira [11], M. Brunella and L.G. Mendes [3], E. Esteves and S. Kleiman [9].

M. Soares in [13] proved the following Theorem for smooth hypersurfaces invariant by foliations on  $\mathbb{P}^n$ .

**Theorem 1.1.** [13] *Let  $\mathcal{F}$  be a holomorphic one dimensional foliation on  $\mathbb{P}^n$  with isolated singularities. If  $V \subset \mathbb{P}^n$  is a smooth hypersurface invariant by  $\mathcal{F}$ , then*

$$\deg(V) \leq \deg(\mathcal{F}) + 1.$$

In [6] M. Corrêa Jr and M. Soares studied the Poincaré problem for foliations on weighted projective planes.

**Theorem 1.2.** *Let  $\mathcal{F}$  be a foliation on the weighted projective plane  $\mathbb{P}(\omega_0, \omega_1, \omega_2)$  such that  $\text{Sing}(\mathcal{F}) \cap \text{Sing}(\mathbb{P}(\omega_0, \omega_1, \omega_2)) = \emptyset$ . If  $S$  is a quasi-smooth invariant curve, then*

$$\deg(S) \leq \deg(\mathcal{F}) + \omega_0 + \omega_1 + \omega_2 - 2.$$

In this work, we give a bounding of degree of quasi-smooth hypersurfaces which are invariant by a one dimensional holomorphic foliation of a given degree on a weighted projective space. Throughout this paper,  $\mathbb{P}(\omega)$  will denote the weighted projective space of dimension  $n$  and weights  $(\omega_0, \dots, \omega_n)$ .

We prove the following theorem.

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**Theorem 1.3.** *Let  $\mathcal{F}$  be a holomorphic one dimensional foliation on  $\mathbb{P}(\omega)$  with isolated singularities, let  $V \subset \mathbb{P}(\omega)$  be a quasi-smooth hypersurface invariant by  $\mathcal{F}$ .*

(i) *If  $n = 2$ , then*

$$\deg(V) \leq \deg(\mathcal{F}) + \omega_0 + \omega_1 + \omega_2 - 2;$$

(ii) *if  $n \geq 3$  and  $\deg(\mathcal{F}) \geq \omega_0 + \dots + \omega_n + 1$ , then*

$$\deg(V) < \deg(\mathcal{F}) + \alpha_n(\omega_0 + \dots + \omega_n) - 1,$$

$$\text{where } \alpha_n = \begin{cases} \text{the positive root of } R_n(x) := x(x+1)^n - 2 = 0 & \text{if } n \text{ is odd} \\ \alpha_{n-1} & \text{if } n \text{ is even} \end{cases}$$

This result say us that the hypothesis  $\text{Sing}(\mathcal{F}) \cap \text{Sing}(\mathbb{P}(\omega)) = \emptyset$  is not necessary. Therefore, the Theorem 1.2 is improved in the case  $n = 2$  and generalized whenever the condition  $\deg(\mathcal{F}) \geq \omega_0 + \dots + \omega_n + 1$  holds.

**Remark 1.4.** *By a direct computation, we can calculate the first values of  $\alpha_n$  with 4 decimal places:*

$n$	3	5	7	9	11	13	15	17	19
$\alpha_n$	0.5436	0.3880	0.3069	0.2563	0.2214	0.1957	0.1759	0.1601	0.1471

*In addition, since*

$$R_n\left(\frac{\ln n - \ln \ln n}{n}\right) < \frac{\ln n - \ln \ln n}{n} \exp(\ln n - \ln \ln n) - 2 = \frac{\ln n - \ln \ln n}{\ln n} - 2 < -1$$

*and for any constant  $\epsilon > 0$ , and all  $n \gg 0$*

$$\begin{aligned} R_n\left(\frac{\ln n - (1 - \epsilon) \ln \ln n}{n}\right) &\approx \frac{\ln n - (1 - \epsilon) \ln \ln n}{n} \exp(\ln n - (1 - \epsilon) \ln \ln n) - 2 \\ &= (\ln n)^\epsilon \left(1 - \frac{\ln n - \ln \ln n}{\ln n}\right) - 2 \gg 0. \end{aligned}$$

*Using that  $R_n(x)$  is an increasing function in  $\mathbb{R}^+$ , it follows that*

$$\frac{\ln n - \ln \ln n}{n} < \alpha_n < \frac{\ln n - (1 - \epsilon) \ln \ln n}{n} \quad \text{for all } n \gg 0.$$

*In general, from the fact that  $\frac{2}{n+1} < \frac{\ln n - \ln \ln n}{n}$  for all  $n \geq 21$ , we have that*  
 $\max\left\{\frac{2}{n+1}, \frac{\ln n - \ln \ln n}{n}\right\} < \alpha_n < \frac{\ln 2n}{n}$  *for all  $n \geq 3$ .*

Let us give a family of examples of holomorphic foliations satisfying the conditions of Theorem 1.3:

Let  $a_0, b_0, a_1, \dots, a_n, b_n$  be positive integers, without common factor in pairs and such that

$$\xi := a_0 + b_0 = \dots = a_n + b_n.$$

and consider the well formed weighted projective space  $\mathbb{P}^{2n+1}(a_0, b_0, \dots, a_n, b_n)$ .

Let  $\mathcal{F}$  be the holomorphic foliation on  $\mathbb{P}^{2n+1}(a_0, b_0, \dots, a_n, b_n)$ , induced by the quasi-homogeneous vector field

$$Z = \sum_{k=0}^n \left( \beta_k Y_k^{\beta_k-1} \frac{\partial}{\partial X_k} - \alpha_k X_k^{\alpha_k-1} \frac{\partial}{\partial Y_k} \right),$$

where the  $\alpha_k, \beta_k \in \mathbb{N}$  satisfy the following relation

$$\zeta = a_k \alpha_k = b_k \beta_k \quad \text{for all } k = 0, \dots, n.$$

A quasi-smooth hypersurface on  $\mathbb{P}^{2n+1}(a_0, b_0, \dots, a_n, b_n)$  of degree  $\zeta$  given by

$$V = \left\{ \sum_{k=0}^n (X_k^{\alpha_k} + Y_k^{\beta_k}) = 0 \right\}.$$

We can see that  $V$  is invariant by  $\mathcal{F}$  and  $\deg(\mathcal{F}) = \zeta - \xi + 1$ . Moreover, since  $a_i$  and  $b_i$  divide  $\zeta$ , it follows that  $\zeta \geq a_0 b_0 \cdots a_n b_n \gg (n+2)\xi$  and

$$\deg(\mathcal{F}) = \zeta - \xi + 1 \geq (n+1)\xi + 1 = 1 + \sum_{j=0}^n a_j + b_j.$$

So, the hypothesis of Theorem 1.3 is satisfied.

Finally, we have

$$\deg(V) - \deg(\mathcal{F}) = \xi - 1 = \frac{1}{n+1} \left( \sum_{j=0}^n a_j + b_j \right) - 1 < \alpha_{2n+1} \left( \sum_{j=0}^n a_j + b_j \right) - 1.$$

We can construct a similar foliation on even dimensional weighted projective spaces  $\mathbb{P}^{2n+2}(a_0, b_0, \dots, a_n, b_n, a_{n+1})$  where  $\xi = a_k + b_k$  for all  $k = 0, \dots, n$ .

Let suppose that  $\zeta = a_k \alpha_k = b_k \beta_k = a_{n+1} \alpha_{n+1}$  for all  $k = 0, \dots, n$  and consider the vector field  $Z$  in the previous example. Thus, the quasi-smooth hypersurface on  $\mathbb{P}^{2n+2}(a_0, b_0, \dots, a_n, b_n, a_{n+1})$  of degree  $\zeta$  given by

$$V = \left\{ \sum_{k=0}^n (X_k^{\alpha_k} + Y_k^{\beta_k}) + X_{n+1}^{\alpha_{n+1}} = 0 \right\}$$

is invariant by  $Z$  and therefore we obtain the same conclusions.

## 2. WEIGHTED PROJECTIVE FOLIATIONS

Let  $\omega_0, \dots, \omega_n$  be integers  $\geq 1$ . Consider the  $\mathbb{C}^*$ -action on  $\mathbb{C}^{n+1} \setminus \{0\}$  given by

$$\lambda \cdot (z_0, \dots, z_n) = (\lambda^{\omega_0} z_0, \dots, \lambda^{\omega_n} z_n),$$

where  $\lambda \in \mathbb{C}^*$  and  $(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ . The *weighted projective space of type*  $(\omega_0, \dots, \omega_n)$  is the quotient space  $\mathbb{P}(\omega_0, \dots, \omega_n) = (\mathbb{C}^{n+1} \setminus \{0\} / \sim)$ , induced by the action above. We will abbreviate  $\mathbb{P}(\omega_0, \dots, \omega_n) := \mathbb{P}(\omega)$ .

Consider the open  $\mathcal{U}_i = \{[z_0 : \dots : z_n] \in \mathbb{P}(\omega_0, \dots, \omega_n); z_i \neq 0\} \subset \mathbb{P}(\omega_0, \dots, \omega_n)$ , with  $i = 0, 1, \dots, n$ . Let  $\mu_{\omega_i} \subset \mathbb{C}^*$  be the subgroup of  $\omega_i$ -th roots of unity. We can define the homeomorphisms  $\phi_i : \mathcal{U}_i \rightarrow \mathbb{C}^n / \mu_{\omega_i}$ , by

$$\phi_i([z_0 : \dots : z_n]) = \left( \frac{z_0}{z_i^{\omega_0/\omega_i}}, \dots, \frac{\hat{z}_i}{z_i}, \dots, \frac{z_n}{z_i^{\omega_n/\omega_i}} \right)_{\omega_i},$$

where the symbol “ $\hat{\phantom{x}}$ ” means omission and  $(\cdot)_{\omega_i}$  is a  $\omega_i$ -conjugacy class in  $\mathbb{C}^n / \mu_{\omega_i}$  with  $\mu_{\omega_i}$  acting on  $\mathbb{C}^n$  by

$$\lambda \cdot (z_0, \dots, \hat{z}_i, \dots, z_n) = (\lambda^{\omega_0} z_0, \dots, \hat{z}_i, \dots, \lambda^{\omega_n} z_n), \lambda \in \mu_{\omega_i}.$$

On  $\phi_i(\mathcal{U}_i \cap \mathcal{U}_j) \subset \mathbb{C}^n / \mu_{\omega_i}$  we have the transitions maps ( $j < i$ )

$$\phi_i \circ \phi_j^{-1}((z_0, \dots, \hat{z}_i, \dots, z_n)_{\omega_i}) = \left( \frac{z_0}{z_j^{\omega_0/\omega_j}}, \dots, \frac{\hat{z}_j}{z_j}, \dots, \frac{1}{z_j^{\omega_i/\omega_j}}, \dots, \frac{z_n}{z_j^{\omega_n/\omega_j}} \right)_{\omega_j}.$$

**2.1. Line bundles on  $\mathbb{P}(\omega)$  and quasi-homogeneous hypersurface.** Let  $d \in \mathbb{Z}$ . Consider the  $\mathbb{C}^*$ -action  $\zeta_d$  on  $\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}$  given by

$$\begin{aligned} \zeta_d : \mathbb{C}^* \times \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C} &\longrightarrow \mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C} \\ (\lambda, (z_0, \dots, z_n), t) &\longmapsto ((\lambda^{\omega_0} z_0, \dots, \lambda^{\omega_n} z_n), \lambda^d t). \end{aligned}$$

We denote the quotient space induced by the action  $\zeta_d$  by

$$\mathcal{O}_{\mathbb{P}(\omega)}(d) := (\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}) / \sim \zeta_d.$$

The space  $\mathcal{O}_{\mathbb{P}(\omega)}(d)$  is a line orbibundle on  $\mathbb{P}(\omega)$ . It is possible to show that the Picard group of  $\mathbb{P}(\omega)$  is generated by  $\mathcal{O}_{\mathbb{P}(\omega)}(1)$ , i.e.

$$\text{Pic}(\mathbb{P}(\omega)) := \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}(\omega)}(1).$$

As usual we set  $\mathcal{O}_{\mathbb{P}(\omega)}(1)^{\otimes d} := \mathcal{O}_{\mathbb{P}(\omega)}(d)$  for  $d \in \mathbb{Z}$ . Moreover, we have the identification ( $d \geq 0$ )

$$H^0(\mathbb{P}(\omega), \mathcal{O}_{\mathbb{P}(\omega)}(d)) = \bigoplus_{\omega_0 k_0 + \dots + \omega_n k_n = d} \mathbb{C} \cdot (z_0^{k_0} \dots z_n^{k_n}).$$

Thus, the global sections of  $\mathcal{O}_{\mathbb{P}(\omega)}(d)$  can be identify, in homogeneous coordinates, with quasi-homogeneous polynomials of degree equal to  $d$ .

Thus, a quasi-homogeneous hypersurface  $V$  on  $\mathbb{P}(\omega)$ , of quasi-homogeneity degree  $d_0$ , is given by  $V = \{f = 0\}$ , where  $f \in H^0(\mathbb{P}(\omega), \mathcal{O}_{\mathbb{P}(\omega)}(d_0))$ . We say that  $V = \{f = 0\}$  is quasi-smooth if its tangent cone  $\{f = 0\}$  on  $\mathbb{C}^{n+1} - \{0\}$  is smooth.

**2.2. Foliations on  $\mathbb{P}(\omega)$  and quasi-homogeneous vector fields.** A singular one dimensional holomorphic foliation on  $\mathbb{P}(\omega)$ , of degree  $d$ , is given by an element of  $\mathbb{P}H^0(\mathbb{P}(\omega), T\mathbb{P}(\omega) \otimes \mathcal{O}_{\omega}(d-1))$ .

On  $\mathbb{P}(\omega)$  we have an Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\omega)} \xrightarrow{\varsigma} \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}(\omega)}(\omega_i) \longrightarrow T\mathbb{P}(\omega) \longrightarrow 0,$$

where  $\mathcal{O}_{\mathbb{P}(\omega)}$  is the trivial line orbibundle and  $T\mathbb{P}(\omega) = \text{Hom}(\Omega_{\mathbb{P}(\omega)}^1, \mathcal{O}_{\mathbb{P}(\omega)})$  is the tangent orbibundle of  $\mathbb{P}(\omega)$ . The map  $\varsigma$  is given explicitly by  $\varsigma(1) = (\omega_0 z_0, \dots, \omega_n z_n)$  (see [13]).

Now, let  $X$  be a quasi-homogeneous vector field of type  $(\omega_0, \dots, \omega_n)$  and degree  $d$  on  $\mathbb{C}^{n+1}$ , i.e.  $X = \sum_{i=0}^n P_i(z) \frac{\partial}{\partial z_i}$  where each polynomial  $P_i$  satisfies the “weight-homogeneous” relation

$$P_i(\lambda^{\omega_0} z_0, \dots, \lambda^{\omega_n} z_n) = \lambda^{d+\omega_i-1} P_i(z_0, \dots, z_n), \quad \forall i = 1, \dots, n.$$

These vector fields descend well to  $\mathbb{P}(\omega)$ . In fact, tensoring the Euler sequence by  $\mathcal{O}_{\mathbb{P}(\omega)}(d-1)$ , we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\omega)}(d-1) \longrightarrow \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}(\omega)}(d+\omega_i-1) \longrightarrow T\mathbb{P}(\omega) \otimes \mathcal{O}_{\mathbb{P}(\omega)}(d-1) \longrightarrow 0.$$

It follows that a quasi-homogeneous vector field  $X$  induces a foliation  $\mathcal{F}$  of  $\mathbb{P}(\omega)$  and that  $gR_{\omega} + X$  define the same foliation as  $X$ , where  $R_{\omega}$  is the adapted radial vector field  $R_{\omega} = \omega_0 z_0 \frac{\partial}{\partial z_0} + \dots + \omega_n z_n \frac{\partial}{\partial z_n}$ , with  $g$  a quasi-homogeneous polynomial of type  $(\omega_0, \dots, \omega_n)$  and degree  $d-1$ . Therefore, a quasi-homogeneous vector field of type  $(\omega_0, \dots, \omega_n)$  and degree  $d$  on  $\mathbb{C}^{n+1}$  induces a holomorphic foliation on  $\mathbb{P}(\omega)$  given by a global section of  $H^0(\mathbb{P}(\omega), T\mathbb{P}(\omega) \otimes \mathcal{O}_{\omega}(d-1))$ .

We have the following condition on the degree of a foliation

$$d > 1 - \max_{0 \leq i < j \leq n} \{\omega_i + \omega_j\}.$$

In fact, by Bott's Formulae for weighted projective spaces (see [8]), we have that

$$H^0(\mathbb{P}(\omega), T\mathbb{P}(\omega) \otimes \mathcal{O}_\omega(d-1)) \simeq H^0(\mathbb{P}(\omega), \Omega_{\mathbb{P}(\omega)}^{n-1}(\sum_{i=0}^n \omega_i + d-1)) \neq \emptyset,$$

if and only if,  $d-1 > -\max_{0 \leq i \neq j \leq n} \{\omega_i + \omega_j\}$ .

An algebraic hypersurface  $V \subset \mathbb{P}(\omega)$  is invariant by a foliation  $\mathcal{F}$  if  $T_p \mathcal{F}_p \subset T_p V$  for all  $p \in V \setminus \text{Sing}(V) \cup \text{Sing}(\mathcal{F})$ .

### 2.3. Orbifold Milnor numbers and Baum-Bott formula.

**Definition 2.1.** Let  $M$  be a complex orbifold and  $\mathcal{F}$  a singular holomorphic foliation on  $M$ . Let  $p \in M$  be and  $(\tilde{U}, G_p, \varphi)$  an orbifold chart  $U$  of  $p$ , the *orbifold Minor number* of  $\mathcal{F}$  on  $p$  is the rational number

$$\mu_p^{\text{orb}}(\mathcal{F}) = \frac{\mu_{\tilde{p}}(\tilde{\xi})}{|G_p|},$$

where  $\mu_{\tilde{p}}(\tilde{\xi})$  is the milnor number of the local lift  $\tilde{\xi}$  on  $\tilde{p}$  of a the vector field  $\xi$  tangent to  $\mathcal{F}$  on  $\tilde{U}/G_p$ .

We will use the following Baum-Bott theorem for orbifolds due to M. Corrêa, A. M. Rodríguez, M. G. Soares [7].

**Theorem 2.2.** *Let  $M$  be a compact complex orbifold, of dimension  $n$ , and  $\mathcal{F}$  a singular holomorphic foliation on  $M$  induced by a global section of  $TM \otimes L$ , with isolated singularities. Then*

$$\int_M^{\text{orb}} c_n(TM \otimes L) = \sum_{p \in \text{Sing}(\mathcal{F})} \mu_p^{\text{orb}}(\mathcal{F}).$$

On Weighted projective spaces and quasi-homogeneous and quasi-smooth hypersurfaces we have the following.

**Corollary 2.3.** [7] *Let  $\mathcal{F}$  be a foliation of degree  $d$  on  $\mathbb{P}(\omega)$ . Then*

$$\sum_{p \in \text{Sing}(\mathcal{F})} \mu_p(\mathcal{F})^{\text{orb}} = \frac{1}{\omega_0 \cdots \omega_n} \sum_{j=0}^n (d-1)^{n-j} \sigma_j(\omega),$$

where  $\sigma_k(\omega)$  denotes the  $k$ -th elementary symmetric function.

**Corollary 2.4.** *Let  $V$  be a quasi-homogeneous and quasi-smooth hypersurface, of degree  $d_0$ , invariant by a holomorphic foliations  $\mathcal{F}$  of degree  $d$ . Then*

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap V} \mu_p(\mathcal{F})^{\text{orb}} = \frac{1}{\omega_0 \cdots \omega_n} \sum_{j=0}^{n-1} \left[ \sum_{k=0}^j (-1)^k \sigma_{j-k}(\omega) d_0^{k+1} \right] (d-1)^{n-1-j},$$

where  $\sigma_\ell(\omega)$  denotes the  $\ell$ -th elementary symmetric function.

*Proof.* It follows from theorem 2.2 that

$$\sum_{p \in \text{Sing}(\mathcal{F}) \cap V} \mu_p(\mathcal{F})^{\text{orb}} = \int_V^{\text{orb}} C_{n-1}(TV \otimes \mathcal{O}_\omega(d-1)|_V).$$

In order to calculate this integral, consider the Euler sequence

$$0 \longrightarrow TV \longrightarrow T\mathbb{P}_\omega^n|_V \longrightarrow N_V \longrightarrow 0,$$

where  $N_V$  is the normal bundle. Then

$$C(\mathbb{P}_\omega^n) = C(V)C(N_V)$$

and

$$(1) \quad C_j(V) = C_j(\mathbb{P}_\omega^n) - C_{j-1}(V)C_1(N_V), \quad 1 \leq j \leq n-1.$$

Moreover, by using the Euler formula we get

$$(2) \quad C_j(\mathbb{P}_\omega^n) = P_j(\omega)C_1(\mathcal{O}_\omega(1))^j.$$

On the other hand, since

$$N_V = \mathcal{O}_{\mathbb{P}_\omega^n}(V)|_V = \mathcal{O}_\omega(d_0)|_V,$$

we have that

$$(3) \quad C_1(N_V) = d_0 C_1(\mathcal{O}_\omega(1)|_V),$$

and replacing (2) and (3) in (1), we obtain

$$C_j(V) = \left( \sum_{k=0}^j (-1)^k \sigma_j(\omega) d_0^k \right) \mathcal{O}_\omega(1)^j, \quad 0 \leq j \leq n-1.$$

Therefore

$$\begin{aligned} & C_{n-1}(TV \otimes \mathcal{O}_\omega(d-1)|_V) \\ &= \sum_{j=0}^{n-1} C_{n-1-j}(V) C_1(\mathcal{O}_\omega(d-1))^j \\ &= \sum_{j=0}^{n-1} C_{n-1-j}(V) C_1(\mathcal{O}_\omega(1))^j (d-1)^j \\ (4) \quad &= \sum_{j=0}^{n-1} \left[ \left( \sum_{k=0}^j (-1)^k \sigma_{j-k}(\omega) d_0^k \right) C_1(\mathcal{O}_\omega(1))^{n-1} \right] (d-1)^{n-1-j}. \end{aligned}$$

By Satake-Poincaré duality, since  $C_1(\mathcal{O}_\omega(d_0))$  is the Satake-Poincaré dual of  $V$ , we obtain

$$\begin{aligned} \int_V^{orb} C_1(\mathcal{O}_\omega(1))^{n-1} &= \int_{\mathbb{P}_\omega^n}^{orb} C_1(\mathcal{O}_\omega(1))^{n-1} \wedge C_1(\mathcal{O}_\omega(d_0)) \\ &= d_0 \int_{\mathbb{P}_\omega^n}^{orb} C_1(\mathcal{O}_\omega(1))^n = \frac{d_0}{\omega_0 \cdots \omega_n}. \end{aligned}$$

The last integral is calculated in [10] and [7]. Finally, the result follows from the equation (4), integrating on the variety  $V$ .  $\square$

## 3. PROOF OF THEOREM

Denote by  $d_0 = \deg(V)$  and  $d = \deg(\mathcal{F})$ . It follows from Corollary 2.4 that

$$(5) \quad \sum_{j=0}^{n-1} \left[ \sum_{k=0}^j (-1)^k \sigma_{j-k}(w) d_0^{k+1} \right] (d-1)^{n-1-j} = (w_0 \cdots w_n) \sum_{p \in V / \xi|_V(p)=0} \mathcal{I}_p(\xi|_V) \geq 0.$$

Also, by Corollary 2.3, we have

$$(6) \quad \sum_{k=0}^n (d-1)^{n-k} \sigma_k(w) - \sum_{j=0}^{n-1} \left[ \sum_{k=0}^j (-1)^k \sigma_{j-k}(w) d_0^{k+1} \right] (d-1)^{n-1-j} \\ = (w_0 \cdots w_n) \left( \sum_{p \in \mathbb{P}_w^n / \xi(p)=0} \mathcal{I}_p(\xi) - \sum_{p \in V / \xi|_V(p)=0} \mathcal{I}_p(\xi|_V) \right) \geq 0.$$

Firstly, we prove for  $n = 2$ . It follows from equation (5) that

$$-d_0^2 + (\sigma_1(w) + d - 1)d_0 = (w_0 w_1 w_2) \sum_{p \in V / \xi|_V(p)=0} \mathcal{I}_p(\xi|_V) \geq 0.$$

Then  $0 < d_0 \leq \sigma_1(w) + d - 1$ . We have that  $V \cap \text{Sing}(\mathcal{F}) \neq \emptyset$ . In fact, suppose that  $V \cap \text{Sing}(\mathcal{F}) = \emptyset$ . In this case  $V$  is a leaf of the foliation and we have the isomorphism of normal orbibundles  $\mathcal{N}_{\mathcal{F}}|_V \simeq N_{V|\mathbb{P}_\omega^2} = \mathcal{O}_\omega(d_0)|_V$ . Since the curve  $V$  is a orbifold and  $V \cap \text{Sing}(\mathcal{F}) = \emptyset$ , it follows from [7] that  $\deg(\mathcal{N}_{\mathcal{F}}|_V) = 0$ . But this is absurd since

$$\deg(\mathcal{N}_{\mathcal{F}}|_V) = \deg(\mathcal{O}_\omega(d_0)|_V) = [\mathcal{O}_\omega(d_0)] \cdot V = [\mathcal{O}_\omega(d_0)] \cdot [\mathcal{O}_\omega(d_0)] = d_0^2 \neq 0.$$

Observe that  $V \cap \text{Sing}(\mathcal{F}) \neq \emptyset$  implied that  $d_0 \neq \sigma_1(w) + d - 1$ , thus  $d_0 \leq \sigma_1(w) + d - 2$  and this proves the Theorem for  $n = 2$ .

For  $n \geq 3$ , in order to prove that  $d_0 \leq d - 1 + \alpha_n \sigma_1(\omega)$ , consider the polynomial  $\Psi(t) \in \mathbb{Z}[t]$  defined by

$$\Psi(t) = \sum_{j=0}^{n-1} \left( \sum_{k=0}^j (-1)^k \sigma_{j-k}(\omega) t^{k+1} \right) (d-1)^{n-1-j},$$

and define  $\Omega_n(t)$  as

$$\Omega_n(t) = \begin{cases} \frac{d}{dt} \left( \frac{\Psi(t)}{t} \right), & \text{if } n \text{ is even} \\ \frac{d}{dt} \Psi(t), & \text{if } n \text{ is odd.} \end{cases}$$

We claim that  $\Omega_n(t) \neq 0$  for all  $t \geq 0$  and  $d \gg 0$ . In order to prove that, for each  $m$  positive integer, we define

$$P_m(t) = \sum_{j=0}^m (-1)^j (m+1-j) t^{m-j} \quad \text{and} \quad Q_m(t) = P_m(t) - P_{m-1}(t)$$

where  $P_0(t) := 1$ .

**Lemma 3.1.** *Let  $m$  be a positive integer. If  $m$  is even then the polynomial  $Q_m(t)$  is positive for all  $t \geq 0$ .*

*Proof.* The result is equivalent to prove that the polynomial

$$F(t) = (t+1)^2 Q_m(t) = (m+1)t^{m+2} + 2t^{m+1} - (m+1)t^m + 2$$

is positive for all  $t \geq 0$ . Observe that  $F(0) = 2$  and  $F(t) \geq 4$  for all  $t \geq 1$ , then it is enough to prove that  $F(t)$  is positive in  $[0, 1]$ . Since

$$F'(t) = (m+1)t^{m-1}((m+2)t^2 + 2t - m),$$

the critical points of  $Q(t)$  are  $t = -1$ ,  $t = 0$  and  $t = \frac{m}{m+2}$ . In addition

$$\begin{aligned} F\left(\frac{m}{m+2}\right) &= \left(\frac{m}{m+2}\right)^m \left( (m+1) \left(\frac{m}{m+2}\right)^2 + 2\left(\frac{m}{m+2}\right) - (m+1) \right) + 2 \\ &= \left(\frac{m}{m+2}\right)^m \left( \frac{-m^2 - 4m - 4}{(m+2)^2} \right) + 2 \\ &> \left( \frac{-m^2 - 4m - 4}{(m+2)^2} \right) + 2 = \frac{4(m+1)}{(m+2)^2} > 0. \end{aligned}$$

Therefore  $Q_m(t) \geq \frac{4(m+1)}{(m+2)^2(t+1)^2} > 0$  for all  $t \geq 0$ .

**Lemma 3.2.** *For all  $k \geq 1$  we have  $\frac{\sigma_{k+1}(\omega)}{\sigma_k(\omega)} < \frac{\sigma_1(\omega)}{k+1}$ .*

*Proof.* Let  $\check{\omega}_I$  be the tuple  $(\omega_1, \dots, \omega_n)$  omitting the coordinates  $\omega_i$ 's with  $i \in I$ . Observe that

$$\begin{aligned} \sigma_k(\omega)\sigma_1(\omega) &= \sum_{i=1}^n \omega_i \sigma_k(\omega) = \sum_{i=1}^n \omega_i (\omega_i \sigma_{k-1}(\check{\omega}_i) + \sigma_k(\check{\omega}_i)) \\ &= \sum_{i=1}^n \omega_i^2 \sigma_{k-1}(\check{\omega}_i) + \sum_{i=1}^n \omega_i \sigma_k(\check{\omega}_i) \\ &= \sum_{i=1}^n \omega_i^2 \sigma_{k-1}(\check{\omega}_i) + (k+1) \sigma_{k+1}(\omega). \end{aligned}$$

Thus, we get that

$$\frac{\sigma_{k+1}(\omega)}{\sigma_k(\omega)} = \frac{\sigma_1(\omega)}{k+1} - \frac{\sum_{i=1}^n \omega_i^2 \sigma_{k-1}(\check{\omega}_i)}{(k+1)\sigma_k(\omega)} < \frac{\sigma_1(\omega)}{k+1}. \quad \square$$

**Proposition 3.3.** *Suppose that  $d \geq \sigma_1(\omega) + 1$ , then the polynomial  $(-1)^{n-1} \Omega_n(t)$  is positive for all  $t \geq 0$ .*



*Proof.* : First, we consider the case when  $n$  is odd. We have that

$$\begin{aligned}
\Omega_n(t) &= \sum_{j=0}^{n-1} \left( \sum_{k=0}^j (-1)^k \sigma_{j-k}(\omega) (k+1) t^k \right) (d-1)^{n-1-j} \\
&= \sum_{l=0}^{n-1} \sigma_l(\omega) \left( \sum_{k=0}^{n-1-l} (-1)^k (k+1) t^k (d-1)^{n-1-k-l} \right) \\
&= \sum_{l=0}^{n-1} \sigma_l(\omega) (d-1)^{n-1-l} \left( \sum_{k=0}^{n-1-l} (-1)^k (k+1) s^k \right) \\
&= \sum_{l=0}^{n-1} \sigma_l(\omega) (d-1)^{n-1-l} (-1)^{n-1-l} P_{n-1-l}(s)
\end{aligned}$$

where  $s = \frac{t}{d-1}$ .

We can write  $\Omega_n(t) - \sigma_{n-1}(\omega)$  as

$$= \sum_{j=0}^{\frac{n-3}{2}} (d-1)^{n-2-2j} (\sigma_{2j}(\omega)(d-1)P_{n-1-2j}(s) - \sigma_{2j+1}P_{n-2-2j}(s)).$$

On the other hand, by Lemma 3.2 and the hypothesis, we know that

$$\sigma_{2l}(\omega)(d-1) \geq \sigma_{2l}(\omega)\sigma_1(\omega) \geq \sigma_{2l+1}(\omega).$$

It follows that

$$\begin{aligned}
\sigma_{2j}(\omega)(d-1)P_{n-1-2j}(s) - \sigma_{2j+1}P_{n-2-2j}(s) &\geq \\
&\geq \sigma_{2j}(\omega)(d-1)P_{n-1-2j}(s) - \sigma_{2j+1}|P_{n-2-2j}(s)| \\
&\geq \sigma_{2j}(\omega)(d-1)(P_{n-1-2j}(s) - |P_{n-2-2j}(s)|) \\
&\geq \sigma_{2j}(\omega)(d-1) \min \{ (n-2j)s^{n-1-2j}, Q_{n-1-2j}(s) \} \\
&> 0.
\end{aligned}$$

This last inequality follows from Lemma 3.1. Therefore  $\Omega_n(t) > \sigma_{n-1}(\omega)$  for all  $t \geq 0$ .

In the case  $n$  even, similarly we obtain that

$$\begin{aligned}
\Omega_n(t) &= \sum_{l=0}^{n-1} \sigma_l(\omega)(d-1)^{n-2-l} (-1)^{n-1-l} P_{n-2-l}(s) \\
&= \sum_{j=0}^{\frac{n-2}{2}} (d-1)^{n-2-2j} (-\sigma_{2j}(\omega)(d-1)P_{n-2-2j}(s) + \sigma_{2j+1}P_{n-3-2j}(s)),
\end{aligned}$$

and by the previous argument, we have that each term of this sum is negative, therefore  $\Omega_n(t) < 0$  for all  $t \geq 0$ .  $\square$

Now, in order to finish the proof of Theorem, we have two case to consider:

3.1.  **$n$  odd:** It follows from Proposition 3.3 that  $\Psi'(t) = \Omega_n(t) > 0$  for all  $t \in \mathbb{R}^+$ .

Then,  $\Psi$  is a increasing function and by equation (6),  $\Psi(d_0) \leq \sum_{l=0}^n (d-1)^{n-l} \sigma_l(w)$ .

We claim that  $\Psi(d-1 + \alpha_n \sigma_1(\omega)) > \sum_{l=0}^n (d-1)^{n-l} \sigma_l(w)$ . In fact, since

$$\begin{aligned} \Psi(t) &= \sum_{j=0}^{n-1} \left( \sum_{k=0}^j (-1)^k \sigma_{j-k}(\omega) t^{k+1} \right) (d-1)^{n-1-j} \\ &= \sum_{l=0}^{n-1} \sigma_l(\omega) (d-1)^{n-l} \sum_{k=0}^{n-1-l} (-1)^k \left( \frac{t}{d-1} \right)^{k+1} \\ &= \frac{t}{d-1+t} \sum_{l=0}^{n-1} \sigma_l(\omega) (d-1)^{n-l} \left( 1 - \left( \frac{-t}{d-1} \right)^{n-l} \right) \\ &= \frac{t}{d-1+t} \sum_{l=0}^{n-1} \sigma_l(\omega) ((d-1)^{n-l} - (-t)^{n-l}), \end{aligned}$$

then, putting  $t = d-1 + \alpha_n \sigma_1(\omega)$  and using that  $\sigma_{2k+1}(\omega) < \frac{1}{2k} \cdot \sigma_{2k}(\omega) \sigma_1(\omega)$  for each  $k \geq 1$ , we have that

$$\begin{aligned} -\Psi(t) + \sum_{l=0}^n \sigma_l(w) (d-1)^{n-l} &= \\ &= \sigma_n(\omega) + \frac{1}{d-1+t} \sum_{l=0}^{n-1} \sigma_l(\omega) ((d-1)^{n-l+1} - (-t)^{n-l+1}) \\ &= \sigma_n(\omega) + \sigma_{n-1}(\omega) \frac{(d-1)^2 - t^2}{d-1+t} + \\ &\quad + \frac{1}{d-1+t} \sum_{k=0}^{\frac{n-3}{2}} (\sigma_{2k}(\omega) ((d-1)^{n-2k+1} - t^{n-2k+1}) \\ &\quad \quad \quad + \sigma_{2k+1}(\omega) ((d-1)^{n-2k} + t^{n-2k})) \\ &< \frac{1}{d-1+t} \sum_{k=1}^{\frac{n-3}{2}} \sigma_{2k}(\omega) \left( (d-1)^{n-2k} \left( d-1 + \frac{1}{2k} \sigma_1(\omega) \right) - t^{n-2k} \left( t - \frac{1}{2k} \sigma_1(\omega) \right) \right) \\ &\quad \quad \quad + (d-1)^{n+1} - t^{n+1} + \sigma_1(\omega) (d-1)^n + \sigma_1(\omega) t^n. \end{aligned}$$

Then, in order to conclude the proof, it is enough to show that each term of this summation is less or equal to zero, or equivalently, putting  $s = d-1$ ,  $\sigma = \sigma_1(\omega)$ , we have to prove that

- (I)  $\left( \frac{t}{s} \right)^{n-2k} \geq \frac{s + \frac{1}{2k} \sigma}{t - \frac{1}{2k} \sigma}$  for each  $k = 1, \dots, \frac{n-3}{2}$  and
- (II)  $t^n(t - \sigma) \geq s^n(s + \sigma)$ .

For item (I), the case  $n = 3$  is empty, then we can suppose that  $n \geq 5$ . By Bernoulli's inequality, we have

$$\left( \frac{t}{s} \right)^{n-2k} = \left( 1 + \frac{\alpha_n \sigma}{s} \right)^{n-2k} \geq 1 + \frac{(n-2k) \alpha_n \sigma}{s},$$

so, it is enough to prove that right side of this inequality is greater than

$$\frac{s + \frac{1}{2k}\sigma}{t - \frac{1}{2k}\sigma} = 1 + \frac{(\frac{1}{k} - \alpha_n)\sigma}{s + (\alpha_n - \frac{1}{2k})\sigma}.$$

In fact, if  $n \geq 7$ , then  $\alpha_n \geq \frac{2}{n}$  and

$$\begin{aligned} & (n-2k)\alpha_n \left( s + \left( \alpha_n - \frac{1}{2k} \right) \sigma \right) - \left( \frac{1}{k} - \alpha_n \right) s \\ & > (n-2k)\alpha_n \left( s - \frac{1}{2k}\sigma \right) - \left( \frac{1}{k} - \alpha_n \right) s \\ & > \left( (n-2k)\alpha_n \frac{2k-1}{2k} - \frac{1}{k} + \alpha_n \right) s \\ & > \left( \frac{n}{2}\alpha_n - 1 \right) s > 0, \end{aligned}$$

and when  $n = 5$ , it follows that  $k = 1$  and

$$(n-2k)\alpha_n \left( s + \left( \alpha_n - \frac{1}{2k} \right) \sigma \right) - \left( \frac{1}{k} - \alpha_n \right) s > \left( \frac{5}{2}\alpha_5 + 3\alpha_5^2 \right) s > 0.$$

Finally, we are going to prove (II). Making  $U := \frac{s}{\sigma} > 1$ , observe that

$$\begin{aligned} & t^n(t-\sigma) - s^n(s+\sigma) \\ & = (s + \alpha_n\sigma)^n(s + (\alpha_n - 1)\sigma) - s^{n+1} - \sigma s^n \\ & = \sigma^{n+1} \left( (U + \alpha_n)^n(U + \alpha_n - 1) - U^{n+1} - U^n \right) \\ & = \sigma^{n+1} \left( ((n+1)\alpha_n - 2)U^n + \sum_{j=1}^{n+1} \left( \binom{n}{j} \alpha_n^j + \binom{n}{j-1} \alpha_n^{j-1}(\alpha_n - 1) \right) U^{n-j+1} \right) \\ & = \sigma^{n+1} \left( ((n+1)\alpha_n - 2)U^n + \sum_{j=0}^{n+1} \left( \frac{n+1}{j} \alpha_n - 1 \right) \binom{n}{j-1} \alpha_n^{j-1} U^{n-j+1} \right) \end{aligned}$$

Now, the polynomial

$$F(X) = ((n+1)\alpha_n - 2)X^n + \sum_{j=0}^{n+1} \left( \frac{n+1}{j} \alpha_n - 1 \right) \binom{n}{j-1} \alpha_n^{j-1} X^{n-j+1}$$

and its derivate  $F'(X)$  satisfy that

- the leading coefficient is positive,
- the list of other coefficients is a decreasing sequence, and then it only has one change of sign,
- $F(0) < 0$  and  $F'(0) < 0$ ,
- $F(1) = (1 + \alpha_n)^n \alpha_n - 2 = 0$ .

Then, by Descartes' rule of signs,  $F'(X)$  only has one positive root and that root is in the interval  $(0, 1)$ , thus  $F(X)$  is an increasing function in  $(1, \infty)$ . Therefore

$$t^n(t-\sigma) - s^n(s+\sigma) = \sigma^{n+1} F(U) \geq \sigma^{n+1} F(1) = 0,$$

as we want to prove.  $\square$

**3.2.  $n$  even.** By the equation (5) we know that  $\Psi(d_0) \geq 0$ . Therefore, if we define  $\Phi(t) = \frac{\Psi(t)}{t}$ , then  $\Phi(d_0) \geq 0$  and from Proposition 3.3 we have that  $\Phi(t)$  is a decreasing function.

We claim that  $\Phi(d - 1 + \alpha_n \sigma_1(\omega)) < 0$ , and therefore  $d_0 < d - 1 + \alpha_n \sigma_1(\omega)$ . In fact, following the same procedure using previously, we have that

$$\begin{aligned} \Phi(t) &= \frac{1}{d-1+t} \sum_{l=0}^{n-1} \sigma_l(\omega) ((d-1)^{n-l} - (-t)^{n-l}) \\ &= \frac{1}{d-1+t} \sum_{k=0}^{\frac{n-2}{2}} (\sigma_{2k}(\omega) ((d-1)^{n-2k+1} - t^{n-2k+1}) \\ &\quad + \sigma_{2k+1}(\omega) ((d-1)^{n-2k} + t^{n-2k})) \end{aligned}$$

From here, the same argument of the case odd works. □

## REFERENCES

- [1] P. Baum, R. Bott: *Singularities of holomorphic foliations*. J. Differential Geom. **7** (1972), 279-432.
- [2] M. Brunella. *Some remarks on indices of holomorphic vector fields*; Publicacions Mathematiques **41**: 527-544, 1997.
- [3] M. Brunella and L. G. Mendes, *Bounding the degree of solutions to Pfaff equations*, Publ. Mat. **44** (2000), 593-604.
- [4] M.M. Carnicer, *The Poincaré problem in the nondicritical case*, Ann. of Math. (2) **140** (1994), 289-294.
- [5] D. Cerveau and A. Lins Neto, *Holomorphic foliations in  $\mathbb{P}^2$  having an invariant algebraic curve*, Ann. Inst. Fourier **41** (1991), 883-903.
- [6] M. Corrêa JR ; M. G. Soares. *A note on Poincaré problem for quasi-homogeneous foliations*. Proceedings of the American Mathematical Society, **140**, (2012), no. 9, 3145-3150.
- [7] M. Corrêa Jr ; A. M. Rodríguez ; M. G. Soares. *A Bott type residue formula on complex orbifolds*, pre-print <http://arxiv.org/abs/1412.5450>, (2014)
- [8] I. Dolgachev, *Weighted projective varieties*, Group actions and vector fields, pp. 34-71, Lecture Notes in Mathematics 956, Springer-Verlag, Berlin-Heidelberg, New York (1982).
- [9] E. Esteves and S. Kleiman: *Bounding solutions of Pfaff equations*. Comm. Algebra **31** (2003), 3771-3793.
- [10] E. Mann, *Cohomologie quantique orbifold des espaces projectifs a poids*. J. Algebraic Geom. **17** (2008), 137-166.
- [11] J. V. Pereira, *On the Poincaré problem for foliations of general type*, Math. Ann. **323** (2002), 217-226.
- [12] H. Poincaré, *Sur l'integration algébrique des equations différentielles du premier ordre et du premier degré*, Rend. Circ. Mat. Palermo **5** (1891), 161-191.
- [13] M. G. Soares, *The Poincaré problem for hypersurfaces invariant by one-dimensional foliations*, Inventiones Mathematicae **128** (1997), 495-500.

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